An Advanced Solution Strategy for Inverse Problems in Multibody Dynamics

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MOTIVATION: Inverse Dynamics

- Trajectory planning problems in robotics

Known: Motion of the mass point in x-y-plane
Unknown: Driving force $F$ and moment $M$.

- Virtual test rig in vehicle dynamics

Known: Accelerations or strains at specific points
Unknown: Driving forces $F_1$, $F_2$, $F_3$. 
MOTIVATION: Parameter Identification

Excitations, measured signals → Multibody system with unknown parameters → Identified parameters
CONTENTS

- Considering inverse problems as optimization problems
- What is the “adjoint method”? 
- Application to Multibody Dynamics
- Examples
- Conclusions
Definition of inverse problems as optimization problems
Consider a semi-explicit DAE-system:

State variables: \( x \in \mathbb{R}^n \) \( \dot{x} = f(x, \lambda, u) \) \( \ldots \) state equations

Algebraic variables: \( \lambda \in \mathbb{R}^m \) \( 0 = g(x) \) \( \ldots \) constraint equations

Parameters and/or control variables: \( u \in \mathbb{R}^p \)

System output variables:
\( s_1(x) \ldots s_N(x) \)

Measured data from experiments, which should coincide with system output:
\( \tilde{s}_1(t) \ldots \tilde{s}_N(t) \)

Error:
\[
J = \sum_{i=1}^{N} \int_{0}^{T} (s_i(x) - \tilde{s}_i(t))^2 dt = \int_{0}^{T} h(x, t) dt
\]
Considering inverse problems as optimization problems:

Find a vector of parameters / control signals $u$, such that the error-function / functional $J(u)$ becomes a minimum.

**Advantages**
- A solution is possible also if the inverse problem has no „exact solution“, i.e. if no $u$ exists which reproduces the desired system output exactly.
- Numerical optimization strategies available (Gradient-, Newton-Methods).

**Possible problems**
- Iteration process ends up in a local minimum.
- Measurements (location, measured state) do not provide information of sensitivity with respect to a parameter / control signal.
Main Problem: How to derive the gradient, i.e. the direction $\delta u$ of the steepest descent of $J(u)$?

Most intuitive way – direct derivatives:

1. Compute derivative of $J$ with respect to the parameters $u$

   \[ J = \int_0^T h(x, t)dt + S(x) \bigg|_T \Rightarrow \nabla J = \int_0^T h_x \cdot x_u dt + S_x \cdot x_u \bigg|_T \]

2. Get sensitivities of state variables by direct differentiation of the state and constraint equations with respect to parameters

   \[
   \dot{x} = f(x, \lambda, u) \Rightarrow \dot{x}_u = f_x \cdot x_u + f_\lambda \cdot \lambda_u + f_u \\
   0 = g(x) \Rightarrow 0 = g_x \cdot x_u
   \]

Problems

- Requires the solutions of $\rho$ (time-variant) differential algebraic systems
- Not applicable if $u=u(t)$ is a control signal unless the signal is discretized
An alternative approach: 
The adjoint method


The basic idea:

1. Add zero terms (system equations) multiplied with the adjoint variables $y, \mu$ to the cost functional:

$$J(u) = \int_0^T h(x,t) dt$$

$$\dot{x} = f(x,\lambda,u), \quad 0 = g(x)$$

$$J = \int_0^T \{h(x,t) + y \cdot [f(x,\lambda,u) - \dot{x}] + \mu \cdot g(x)\} dt,$$

2. Define the **Hamiltonian**: $H(x,y,\lambda,\mu,u,t) = h(x,t) + y \cdot f(x,\lambda,u) + \mu \cdot g(x)$

$$J = \int_0^T (H - y \cdot \dot{x}) dt \quad \text{... augmented error function(al)}$$
3. Compute the variation \( \delta J \) of the functional \( J(u) \) due to a control variation \( \delta u \):

\[
J = \int_0^T (H - y \cdot \dot{x}) dt
\]

\[
\delta J = \int_0^T (H_u \cdot \delta u + H_x \cdot \delta x + H_\lambda \cdot \delta \lambda - y \cdot \delta \dot{x}) dt
\]

\[
= \int_0^T (H_u \cdot \delta u + H_x \cdot \delta x + H_\lambda \cdot \delta \lambda + \dot{y} \cdot \delta x) dt - y \cdot \delta x \bigg|_T + y \cdot \delta x \bigg|_0
\]

Choose \( y(t) \) and \( \mu(t) \), that

\[
\dot{y} = -H_x
\]

\[
0 = H_\lambda
\]

\[
0 = y(T)
\]

\[
\delta J = \int_0^T H_u \cdot \delta u \ dt
\]
Summary: The adjoint method for computing the gradient of $J(u)$

**Step 1:** Build the Hamiltonian from the system equations and $h(x,t)$

$$H(x, y, \lambda, \mu, u, t) = h(x, t) + y \cdot f(x, \lambda, u) + \mu \cdot g(x)$$

**Step 2:** For given parameters $u$ and/or control variables $u(t)$ solve the system equations (DAE) forward in time

$$\dot{x} = f(x, \lambda, u), \quad 0 = g(x), \quad x(0) = x_0$$

**Step 3:** Solve the DAE-system for the adjoint variables backwards in time

$$\dot{y} = -H_x, \quad 0 = H_\lambda, \quad y(T) = 0$$

**Step 4:** Since $\delta J = \int_0^T H_u \cdot \delta u \, dt$ compute an update of $u$:

Parameter identification: $\delta u = -\kappa \int_0^T H_u \, dt$

Optimal control problem: $\delta u(t) = -\kappa H_u(t)$

... direction of the steepest descent
Summary: The adjoint method for computing the gradient of $J(u)$

Advantages

- Requires only the solutions of one (time-variant) differential algebraic system (adjoint equations) backwards in time
- The method is also applicable if $u=u(t)$ is a control signal

Problem

- Adjoint equations are difficult to obtain (requires the system Jacobian at every time instant along a forward simulation)

But:

- Redundant description of the system equations facilitates the generation of the adjoint equations
Application to Multibody Dynamics
**FreeDyn**: A new multibody software developed in Wels

**Features:**
- Redundant formulation of the rotations based on **Euler parameters**
- Time integration with HHT-algorithm (without index reduction)
- Floating reference frame formulation for flexible bodies
- Innovative user interface
- Implementation of the system’s adjoint equations for solving optimization problems
Research areas:
Multibody dynamics in general, inverse dynamics and parameter identification, model reduction, contact mechanics
Equations of motion for multibody systems in descriptor form:

\[
\begin{align*}
\dot{q} &= v \\
\mathbf{M}(q,u)\dot{v} &= f(q,v,u,t) - \mathbf{C}^T(q,u)\lambda \\
0 &= c(q,u)
\end{align*}
\]

Properties:
- Nonlinear index three DAE for \( q(t), v(t), \lambda(t) \)
- Constraint Jacobian: \( \mathbf{C}(q,u) = \mathbf{D}_q c \)
- Lagrange multipliers \( \lambda \) are related to the constraint forces
- Control/parameter may occur in \( \mathbf{M}, c \) and \( f \)

Optimization problem:
\[
J(u) = \int_0^T h(q,v,t)dt \rightarrow \text{Min}
\]
Adjoint equations of motion for multibody system:

\[ \dot{p} = A(t)w + C^T(t)\mu + h_q(t) \]
\[ \frac{d}{dt}(M(t)w) = -p + B(t)w + h_v(t) \]
\[ 0 = C(t)w \]

Properties:
- Linear time-variant DAE with index three for \( p(t), w(t), \mu(t) \)
- Boundary conditions: \( p(T) = 0, w(T) = 0 \)
- The following Jacobian matrices are required along a forward simulation:
  \[ A(t) = D_q(M\dot{\nu} - f - C^T \lambda), \quad B(t) = -D_v f \]

Gradient formula:
(if \( u \) appears only in \( f \))
\[ \delta u(t) = \kappa w^T f_u \quad \text{or} \quad \delta u = \kappa \int_0^T w^T f_u \, dt \]
Some examples
Parameter identification of a cart pendulum

Task
- Identification of stiffness and damping parameters

Excitation $F(t)$
- Pulse function approximated with Fourier series

Measured signal(s) $x(t)$
- No real measurements used, signals out of simulation run with original parameters + noise

Correct parameters
- Linear stiffness $c_c = 50 \text{N/mm}$
- Linear damping cart $d_c = 0.3 \text{Ns/mm}$
- Linear damping pendulum $d_c = 0.1 \text{Ns/mm}$
Parameter identification of a cart pendulum

BFGS-method versus gradient-method

Convergence stiffness parameter cc

Convergence cost function
Parameter identification of a cart pendulum

Influence of the velocity error in the cost functional (BFGS approximation of Hessian)
Identification of a body’s inertia moments

**Known signals:**
- Motion $\vec{r}_A(t)$ of point A (excitation)
- Velocity $\vec{r}_B(t)$ of point B (from measurement)

**Known parameters:**
- Positions of A and B in the body fixed frame 1-2-3
- Total mass and center of mass

**Task:**
- Identify $I_1, I_2, I_3$
Planar overhead crane: Inverse dynamics (optimal control problem)

Task:
- System is actuated by the force $F(t)$ and by the moment $M(t)$
- Load with the mass $m$ should follow a desired trajectory $\bar{x}_m(t), \bar{y}_m(t)$

System description by redundant coordinates: $q = [s, l, x_m, y_m]^T$

Constraint equation: $c(q) = (x_m - s)^2 + y_m^2 - l^2 = 0$

Analytical solution is known for this problem
Planar overhead crane: Inverse dynamics
The inverted pendula

Task:

- Find a control $F(t)$ such that the pendula move to the upper (unstable) position by minimizing the functional

$$S = x_0^2 + \dot{x}_0^2 + (\varphi_1 - \pi)^2 + \dot{\varphi}_1^2 + (\varphi_2 - \pi)^2 + \dot{\varphi}_2^2 \bigg|_{t=T}$$

- Use a redundant description of the system

- Include position constraints in the error functional (penalty function $g$)

$$J = \int_0^T g(x_0) dt + S$$
Conclusions

- Many inverse problems such as parameter identification and trajectory tracking can be considered as optimization problems.
- The adjoint method is an efficient and robust solution strategy for optimization problems with dynamical systems.
- The adjoint equations for multibody systems can be generated with acceptable effort by using redundant formulations and Euler parameters.

Current work

- Implementation of the adjoint method in the MBS-code Freedyn.
- Considering mixed problems (parameter identification + optimal control).