Comparison of CMS, moment matching and balanced truncation based model reduction from a mechanical application engineer’s perspective

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Overview

- Motivation
- Mechanical application engineer’s perspective ???
- General issues
- Brief theoretical introduction
  - CMS
  - Moment Matching
  - Balanced Truncation
- Generic Beam example
- Car body component example
- Conclusion
Motivation

• CMS for model reduction
  • Reliable
  • Available

• Last years: Method from control engineering for reduction of FE models
  • Krylov Subspace Method (KSM)
  • Balanced Truncation (BT)
  • Pushed in some papers: ‘modern methods’
    • BT: Clear defined a priori error bound

• KSM and BT are based on mathematical considerations
CMS is based on mechanical considerations
  • Which method for reduction of metallic structures
    • Specially: Multi Body Dynamics (MBD)
Mechanical application engineer’s perspective?

• Should work to get the job done (on a daily base)
  • Reduction: Black box
    • Just moderate understanding of the theory can be expected
    • Clear and simple to use: Reduction following a ‘To Do’ List
      • No convergence analysis to get number of modes
    • No analysis of full system to get modes for reduced system
  • Reliable
    • Terms of displacement AND stress (fatigue)
  • Computational efficient
  • Available in standard FE software
General issues

• Stress recovery
  • Stress ≠ Displacement
  • Full system response investigated
    • Just Input / Output behavior not enough (DIFFERENT to control engineering)
  • Static important (Static content of solution → mean stress → fatigue)

• Restriction: Linear and metallic structures
  • Conclusions may NOT be valid in the presence of significant viscosity (like large rubber domain) and non linear systems
  • Reduction of conservative system (Viscous damping not realistic - Overestimation)
    • Transferring unrealistic damping into reduced system followed by error analysis makes no practical sense

• Reduced System: Stable and symmetric
  • Important for time integration
  • Symmetry: physically meaningful characteristic

• MBD: Arbitrary connections to surrounding should be possible
General issues – Model reduction

• Original system

\[ \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \]

- \( \mathbf{x} \) contains displacements (n x 1)
- \( \mathbf{f} \) contains external forces (n x 1)
- \( \mathbf{M}, \mathbf{K} \) mass and stiffness matrix (n x n)
- \( n \) number of DOF (let’s say 1e7)

• Corresponding reduced model (of size m ( << n))

\[ \mathbf{\tilde{M}}\ddot{\mathbf{z}} + \mathbf{\tilde{K}}\mathbf{z} = \mathbf{\tilde{f}} \]

- \( \mathbf{z} \) contains generalized DOF (m x 1)
- \( \mathbf{\tilde{f}} \) contains reduced external forces (m x 1)
- \( \mathbf{\tilde{M}}, \mathbf{\tilde{K}} \) reduced mass and stiffness matrix (m x m)
- \( m \) number of considered Ritz vectors

\[ \mathbf{z} = \mathbf{V}\mathbf{x} \]

- \( \mathbf{V} \) contains reduction base (n x m)

\[ \mathbf{\tilde{f}} = \mathbf{V}^T\mathbf{f} \]

Requirement for a ‘good’ reduction procedure: Reduced model should be as accurate as possible within defined bounds.
Component Mode Synthesis - CMS

• Component Mode Synthesis based reduction base

\[ V = \begin{bmatrix} V_S & V_D \end{bmatrix} \]

\( V_D \) Vibration modes (capture Dynamics)
Modes up to defined frequency limit (depends on excitation) regarded

\( V_S \) Static deflection shape (accelerate convergence for arbitrary mounting situation)

For this test: Craig’s method (Fixed boundary CMS)

• No ‘a priori’ error in terms of displacement (or stress) available but error limit for eigenfrequencies and eigenvectors (modes)

• Easy and clear to compute (as long as a frequency limit is known)
Krylov Subspace Methods (KSM)

• Transfer function matrix in frequency domain \((s = \text{complex variable})\)
  \[
  x(s) = H(s) f(s) \quad \quad B_1 \text{ defines input DOF (force)}
  
  H(s) = C_1 s (s^2 M + K)^{-1} B_1 \quad \quad C_1 \text{ defines output DOF (displacement)}
  
  \]

• Idea: Approximation of transfer function matrix with power series
  \[
  H(s) = \sum_{j=0}^{\infty} T_j^{s_0} (s_0 - s)^j \quad \quad T_j^{s_0} \text{ called: } j\text{-th moment}
  
  \]

• Construction of ‘Krylov Subspace’ with \(m\) vectors ‘ensures’ that \(2m\)- moments of
  the reduced system will mach with the non reduced system.

• No ‘a priori’ error in terms of displacement (or stress) available but error
  limit for eigenfrequencies and eigenvectors (modes)
  \[
  \text{• No frequency limit can be given without full model}
  
  \]
Krylov Subspace Methods (KSM)

• Illustration – SISO Example

Taken from: Lohman B., Salimabahrami B., Ordnungsreduktion mittels Krylov-Unterraummethoden, Automatisierungstechnik 52, pp. 30 – 38, 2004
Balanced Truncation (BT)

• Main idea of Balanced Truncation (no a sufficient theory)
  - Compute k frequency response shapes
    \[ (-\omega_i^2 M + K) X_i = B_i \quad 0 < \omega_i < \omega_{Limit} \quad 1 \leq i \leq k \]
  - Collect all shapes in one matrix \( X = [X_1 \ X_2 \ X_3 \ldots \ X_k] \)
  - Apply POD to \( X \) \( \rightarrow \) Resulting POMs equal to BT trial vectors
    - POD determines \( u_1, u_2 \ldots u_m \) so that
      \[
      \max_{u_1, \ldots, u_m \in \mathbb{R}^n} \left[ (x_1^T u_1)^2 + \ldots + (x_{(k \times u)}^T u_1)^2 + \ldots + (x_1^T u_m)^2 + \ldots + (x_{(k \times u)}^T u_m)^2 \right]
      \]
    - s.t. \( u_j^T u_i = \delta_{ij} \)
      - Reduction base: \( V = [u_1 \ u_2 \ldots u_m] \)
      - For conservative systems \( \rightarrow \) Vibration modes dominate spanned space
      - Criteria for selection \( \rightarrow \) Euclidean distance (\( \neq \) eigenfrequency !!)
      - Statics undefined
    - Sometimes claimed: Error bound a priori known
Generic Beam example

Beam structure
A fixed to ground (all the time)
B 2 input / output DOF (lateral and vertical)
C 6 input / output DOF

Variant 1: C is free

Variant 2 (off tuned): C is fixed

• Just Variant 1: Static solution due to axial force acting on C

• Investigations for both variants
  • Static solution due to force acting on B
  • Relative error of eigenfrequencies
  • Relative error of MAC values of modes

• In order to compare fair: 16 trial vectors for each method
  • Based on CMS experience: Evaluation up to 200Hz
Generic Beam example
Variant 1 – Static axial force on tip

Axial deflection of bottom section

No trial vectors at all in dynamical stiff directions \(\rightarrow\) Statics undetermined
Generic Beam example

Variant 1 – free end

\[ \varepsilon_i \] relative error of eigenfrequency (with respect to full system)

\[ \beta_i \] relative error of MAC value of mode \( i \) (with respect to full system)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Freq. [Hz]</th>
<th>CMS</th>
<th>K</th>
<th>BT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \varepsilon_i )</td>
<td>( \beta_i )</td>
<td>( \varepsilon_i )</td>
<td>( \beta_i )</td>
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<td>56.3</td>
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<tr>
<td>Static</td>
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<td>1.7e-9</td>
<td></td>
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</table>

Green – Error below 1e-3
Cyan – Error below 1e-2
Yellow – Error below 1e-1
Red – Error higher as 1e-1

22 trial vectors needed to get that torsion mode
(Order due to HSV different than due to vibration frequencies)

• What happened to the torsion mode in BT??

Worst static solution
Generic Beam example
Variant 1 – free end

- Remember: Trial vectors due to BT (POD) minimize somehow ('dirty explanation') the Euclidean distance between full and reduced system
  - Selection criteria (Hankel Singular Values) correspond with contribution of one mode the this distance ('again dirty')

- Torsion mode does not contribute that much to x as other modes

- BUT: Torsion Mode can have significant impact on stress (e.g. crankshaft)
Generic Beam example
Variant 2 – fixed end

<table>
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<td></td>
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<tr>
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<td>44.4</td>
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<tr>
<td>Static</td>
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</table>

- First torsion mode missing
- Worst static solution.

- More detailed look on static solution of balanced truncation
Generic Beam example
Variant 2 – fixed end: Static solution

Note: CMS solution exact

Note furthermore: If there is no trial vector in a certain direction → Error infinite

Consequently: A priori error estimation for BT just valid for frequencies ≠ 0 !!
Car body component Illustration

- Difference between ‘mode selection by eigenfrequency’ (CMS) or ‘mode selection by Euclidean distance’ (BT)
Car body component Illustration

• Difference between ‘mode selection by eigenfrequency’ (CMS) or ‘mode selection by Euclidean distance’ (BT)

Sequence according to eigenfrequency

Sequence according to Euclidean norm (BT)
Conclusion

<table>
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<th>CMS</th>
<th>Krylov</th>
<th>BT</th>
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<tbody>
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<td>Automatic selection of number of</td>
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<td>✗</td>
<td>✓ - ✗</td>
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<td></td>
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<tr>
<td>Varying boundary conditions</td>
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<td>✗</td>
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<tr>
<td>Stress recovery</td>
<td>✓</td>
<td>✓</td>
<td>?- ✗</td>
</tr>
<tr>
<td>Statics</td>
<td>✓</td>
<td>✓</td>
<td>?- ✗</td>
</tr>
</tbody>
</table>

Model reduction for metallic and linear structures as ‘black box’:

**Nothing better available as good old CMS (and variants)**

**BUT**: CMS with BT modes could be an interesting option when stress is not important and in the presence of a lot of non-relevant local modes (Dynamics of car body)